Periodic waves and solitons of two-photon propagation

1996 J. Phys. A: Math. Gen. 29 4127

(http://iopscience.iop.org/0305-4470/29/14/032)

J. Phys. A: Math. Gen. **29** (1996) 4127–4139. Printed in the UK

Periodic waves and solitons of two-photon propagation

A M Kamchatnov† and F Ginovart‡

† Institute of Spectroscopy, Troitsk, Moscow Region, 142092 Russia
‡ Instituto de Física Teórica, Universidade Estadual Paulista, Rua Pamplona 145, 01405-900 São Paulo, SP, Brazil

Received 14 November 1995, in final form 11 March 1996

Abstract. We study the two-photon propagation (TPP) modelling equations. The one-phase periodic solutions are obtained in an effective form. Their modulation is investigated by means of the Whitham method. The theory developed is applied to the problem of creation of TPP solitons on the sharp front of a long pulse.

1. Introduction

Experimental investigation of two-photon propagation (TPP) solitons is rather difficult because such a soliton propagates on the background of a longer pulse and disappears at one of its ends. More intensive pulses lead to the creation of a nonlinear periodic wave as was shown for stimulated Raman scattering [1]. This poses the problem of describing soliton creation on the front of the pulse. The creation of solitons is caused in this case by the modulation instability which transforms the wavefront into a non-uniform region of nonlinear oscillations.

In this work, we study the TPP modelling equations†. As has already been used for other analogous problems [2–6], the Whitham method is used. To give the full description of the non-uniform region, we have to find the periodic solution in an effective form. The solution of this problem can be obtained by means of modification of the well known finite-band integration method [7] (when the operators of the Lax pair are not self-adjoint). Such a modification was suggested in [8] and has been applied to a number of physically significant integrable equations [2–6, 9–11].

2. Periodic solutions of TPP equations

2.1. Derivation of the periodic solutions

The TPP equations describe the propagation of two waves with frequencies ω_1 and ω_2 and envelope electric fields E_1 and E_2 in a medium with resonance transition at the frequency $\omega_1 + \omega_2$. The equations acquire symmetric form if we introduce the vector $\boldsymbol{S}$ with the components [13]

$$S_1 = E_1^* E_2^* + E_2 E_1 \qquad S_2 = \mathrm{i}(E_1^* E_2^* - E_2 E_1) \qquad S_3 = E_1 E_1^* + E_2 E_2^* \tag{1}$$

† An analogous problem on stimulated Raman scattering will be discussed in a separate publication because of the large number of differences in formulae and final results.

0305-4470/96/144127+13$19.50

and pass from the retarded time $t' = t - x/c$ (x is a space coordinate along which the wave propagates and c is their group velocity) to the variable

$$\tau = k \int_{t_0}^{t} I(t')\,\mathrm{d}t' \tag{2}$$

where $I(t) = E_1 E_1^* - E_2 E_2^*$ is the difference of the two field intensities, k is the coupling constant of the dipole interaction of the fields with the medium. If we also introduce the dimensionless space coordinate ξ and the Bloch vector $\boldsymbol{R}$ describing the state of the medium ($R_\pm = R_1 \pm \mathrm{i}R_2$ correspond to non-diagonal elements of the density matrix and R_3 to the difference of populations of the upper and the lower levels of the molecules), then the TPP equations take the form [13, 14]

$$\begin{aligned} \frac{\partial R_+}{\partial \tau} &= \mathrm{i}(\Delta R_+ S_3 + R_3 S_+) \qquad & \frac{\partial R_3}{\partial \tau} &= \frac{\mathrm{i}}{2}(R_+ S_- - R_- S_+) \\ \frac{\partial S_+}{\partial \xi} &= \mathrm{i}(\Delta S_+ R_3 - S_3 R_+) \qquad & \frac{\partial S_3}{\partial \xi} &= \frac{\mathrm{i}}{2}(S_+ R_- - S_- R_+) \end{aligned} \tag{3}$$

where $S_\pm = S_1 \pm \mathrm{i}S_2$ and Δ is the relative dynamic Stark shift coefficient. The vectors $\boldsymbol{R}$ and $\boldsymbol{S}$ are normalized according to the conditions

$$R_1^2 + R_2^2 + R_3^2 = 1 \qquad -S_1^2 - S_2^2 + S_3^2 = 1. \tag{4}$$

In [13, 14] it was shown that the system (3) is integrable by the inverse scattering transform method which permits one to obtain its soliton as well as multi-soliton [15] solutions. The inverse scattering transform method is based on the possibility of presenting equations (3) as a compatibility condition of two linear systems

$$\frac{\partial \psi}{\partial \tau} = \begin{pmatrix} F & G \\ H & -F \end{pmatrix} \psi \qquad \frac{\partial \psi}{\partial \xi} = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \psi \tag{5}$$

where $\psi = (\psi_1, \psi_2)^{\mathrm{T}}$ is a two-component 'spinor' of solutions of equations (5). The general AKNS scheme [16] leads to the equations (3) if one takes the following coefficients [13, 14]

$$F = -\mathrm{i}\lambda S_3 \qquad G = (\lambda + \sigma) S_+ \qquad H = (\lambda - \sigma) S_- \tag{6}$$

$$A = \frac{\mathrm{i}}{2}\left(\Delta + \frac{1}{2\lambda + \Delta}\right) R_3 \qquad B = -\frac{\lambda + \sigma}{2\lambda + \Delta} R_+ \qquad C = -\frac{\lambda - \sigma}{2\lambda + \Delta} R_- \tag{7}$$

where the parameter σ is connected with Δ according to

$$\sigma^2 = \tfrac{1}{4}(1 + \Delta^2) \tag{8}$$

and λ is an arbitrary spectral parameter.

The systems (5) have two basic solutions, (ψ_1, ψ_2) and (φ_1, φ_2), which can be used to build a vector with the spherical components

$$f = -\tfrac{1}{2}\mathrm{i}(\psi_1 \varphi_2 + \psi_2 \varphi_1) \qquad g = \psi_1 \varphi_1 \qquad h = -\psi_2 \varphi_2 \tag{9}$$

satisfying the following linear systems:

$$\begin{aligned} \partial f/\partial \tau &= -\mathrm{i}Hg + \mathrm{i}Gh \qquad & \partial f/\partial \xi &= -\mathrm{i}Cg + \mathrm{i}Bh \\ \partial g/\partial \tau &= 2\mathrm{i}Gf + 2Fg \qquad & \partial g/\partial \xi &= 2\mathrm{i}Bf + 2Ag \\ \partial h/\partial \tau &= -2\mathrm{i}Hf - 2Fh \qquad & \partial h/\partial \xi &= -2\mathrm{i}Cf - 2Ah. \end{aligned} \tag{10}$$

The length of the vector with components (9),

$$f^2 - gh = P(\lambda) \tag{11}$$

does not depend on τ and ξ. The periodic solution is distinguished by the condition that $P(\lambda)$ be a polynomial in λ [17–19]. The single-phase solution corresponds, as we shall see, to the fourth-degree polynomial

$$P(\lambda) = \prod_{i=1}^{4}(\lambda - \lambda_i) = \lambda^4 - s_1\lambda^3 + s_2\lambda^2 - s_3\lambda + s_4. \tag{12}$$

It is easy to find that the systems (10) with the coefficients (6), (7) are satisfied if we take

$$f = S_3\lambda^2 - f_1\lambda + f_2 \qquad g = (\lambda + \sigma)S_+(\lambda - \mu) \qquad h = (\lambda - \sigma)S_-(\lambda - \mu^*) \tag{13}$$

provided f_1, f_2, μ, μ^* satisfy the conditions

$$\begin{aligned} &2f_1S_3 + (1 - S_3^2)(\mu + \mu^*) = s_1 \\ &2f_1f_2 - (1 - S_3^2)\sigma^2(\mu + \mu^*) = s_3 \\ &f_1^2 + 2f_2S_3 + (1 - S_3^2)(-\sigma^2 + \mu\mu^*) = s_2 \\ &f_2^2 - (1 - S_3^2)\sigma^2\mu\mu^* = s_4 \end{aligned} \tag{14}$$

and the following equations are also fulfilled:

$$\frac{\partial S_3}{\partial\tau} = \mathrm{i}(1 - S_3^2)(\mu - \mu^*) \qquad \frac{\partial S_+}{\partial\tau} = -2\mathrm{i}(f_1 - \mu S_3)S_+ \tag{15}$$

$$R_+S_-\left(\mu^* + \tfrac{1}{2}\Delta\right) = R_-S_+\left(\mu + \tfrac{1}{2}\Delta\right) \qquad f\left(-\tfrac{1}{2}\Delta\right)R_+ + \tfrac{1}{2}\left(\mu + \tfrac{1}{2}\Delta\right)R_3S_+ = 0. \tag{16}$$

If we substitute (13) into (10) and put $\lambda = \mu$, then we obtain the evolution equations for μ:

$$\frac{\partial\mu}{\partial\tau} = -2\mathrm{i}f(\mu) = -2\mathrm{i}\sqrt{P(\mu)} \qquad \frac{\partial\mu}{\partial\xi} = -\frac{R_+}{(2\mu + \Delta)S_+}\frac{\partial\mu}{\partial\tau}. \tag{17}$$

Let us write the relations (16) in the form

$$\frac{R_+}{(\mu + \frac{1}{2}\Delta)S_+} = \frac{R_-}{(\mu^* + \frac{1}{2}\Delta)S_-} = -\frac{R_3}{2f(-\frac{1}{2}\Delta)} = \frac{2}{V} \tag{18}$$

where, as we shall see, V is the nonlinear phase velocity of the wave. From equation (18) we find

$$\frac{1 - R_3^2}{(1 - S_3^2)(\mu + \frac{1}{2}\Delta)(\mu^* + \frac{1}{2}\Delta)} = \frac{R_3^2}{4f^2(-\frac{1}{2}\Delta)} = \frac{4}{V^2}.$$

If we put $\lambda = -\frac{1}{2}\Delta$ in (11), then we have

$$(S_3^2 - 1)\left(\mu + \tfrac{1}{2}\Delta\right)\left(\mu^* + \tfrac{1}{2}\Delta\right) = 4\left[P\left(-\tfrac{1}{2}\Delta\right) - f^2\left(-\tfrac{1}{2}\Delta\right)\right]$$

and, hence, the preceding equation gives

$$V = 4\sqrt{P\left(-\tfrac{1}{2}\Delta\right)}. \tag{19}$$

Thus, μ depends only on the phase

$$W = \tau - \frac{\xi}{V} \qquad \frac{\mathrm{d}\mu}{\mathrm{d}W} = -2\mathrm{i}\sqrt{P(\mu)}. \tag{20}$$

The last equation of the system (3) can also be transformed with the help of (15), (16) into the form

$$\frac{\partial S_3}{\partial\xi} = -\frac{1}{V}\frac{\partial S_3}{\partial\tau}$$

thus S_3 also depends only on the phase W.

With this change of W, the variable μ moves along some curve which defines the contour of integration when one calculates $\mu(W)$ according to (20). Therefore it is convenient to determine this contour explicitly for μ by means of introducing some coordinate parameter along it (see [8]). From equation (14) it seems natural to take S_3 as such a parameter, so that μ is to be expressed as a function of S_3. Then the identity (11) is satisfied automatically. The system (14) actually coincides with the analogous system in [10], so let us use its solution. For f_1 and f_2 we have

$$f_1^2 = \frac{1}{2\sigma^2}\left[\sigma^4 + s_2\sigma^2 + s_4 - \sqrt{P_2(\sigma^2)}\right] \tag{21}$$

where

$$\begin{aligned} P_2(\sigma^2) &= \prod_{i=1}^{4}(\lambda_i^2 - \sigma^2) \\ f_2 &= \frac{s_3 + s_1\sigma^2}{2f_1} - \sigma^2 S_3. \end{aligned} \tag{22}$$

The sign of f_1 is determined by the stability condition of the solution $S_3 = -R_3 = 1$ (see [13]). As we shall see, the choice of positive sign, i.e. $f_1 = +\sqrt{f_1^2}$, leads to the stable solution.

Equations for S_+ in (3) and (15), (16), (22) yield

$$\frac{\partial S_+}{\partial \xi} = -\frac{2\mathrm{i}}{V}\left[4f_1\sigma^2 + \frac{(s_3 + s_1\sigma^2)\Delta}{f_1}\right]S_+ - \frac{1}{V}\frac{\partial S_+}{\partial \tau}$$

that is

$$S_+ = \exp\left\{-\frac{2\mathrm{i}}{V}\left[4f_1\sigma^2 + \frac{(s_3 + s_1\sigma^2)\Delta}{f_1}\right]\xi\right\}\tilde{S}_+ \tag{23}$$

where $\tilde{S}_+$ depends only on the phase W and is determined by the equation

$$\frac{\mathrm{d}\tilde{S}_+}{\mathrm{d}W} = -2\mathrm{i}(f_1 - \mu S_3)\tilde{S}_+. \tag{24}$$

The parameter μ is expressed in terms of S_3 as follows (see [10]):

$$\mu = \frac{s_1 - 2f_1S_3 + 2\mathrm{i}\sqrt{-\sigma^2 R(S_3)}}{2(1 - S_3^2)} \tag{25}$$

where

$$R(\nu) = \nu^4 - \frac{s_3 + s_1\sigma^2}{f_1\sigma^2}\nu^2 + \frac{s_2}{\sigma^2}\nu^2 - \left(\frac{s_1f_1}{\sigma^2} - \frac{s_3 + s_1\sigma^2}{f_1\sigma^2}\right)\nu - \frac{4s_2 - 4f_1^2 - s_1^2 + 4\sigma^2}{4\sigma^2} \tag{26}$$

is the algebraic resolvent of the polynomial $P(\lambda)$ whose zeros ν_i, $i = 1, 2, 3, 4$, are related to the zeros λ_i, $i = 1, 2, 3, 4$, of $P(\lambda)$ by the formulae obtained in [10]:

$$\begin{aligned} \nu_1 = -\frac{1}{4f_1\sigma^2}&[(\lambda_1 - \lambda_3)(\lambda_2' - \lambda_4') + (\lambda_2 - \lambda_4)(\lambda_1' - \lambda_3')]^{-1} \\ &\times\{(\lambda_1 - \lambda_3)[-2(\lambda_1 + \lambda_3)(\lambda_2' - \lambda_4')\sigma^2 \\ &+(\lambda_2\lambda_4' - \lambda_4\lambda_2')((\lambda_1 + \lambda_3)^2 - (\lambda_1' - \lambda_3')^2)] \\ &+(\lambda_2 - \lambda_4)[-2(\lambda_2 + \lambda_4)(\lambda_1' - \lambda_3')\sigma^2 \\ &+(\lambda_1\lambda_3' - \lambda_3\lambda_1')((\lambda_2 + \lambda_4)^2 - (\lambda_2' - \lambda_4')^2)]\} \end{aligned} \tag{27}$$

where

$$\lambda_i' = \sqrt{\lambda_i^2 - \sigma^2}$$

and ν_2 and ν_3 are obtained from ν_1 by means of exchange of indices $3 \leftrightarrow 4$ and $3 \leftrightarrow 2$, respectively, and ν_4 can be obtained from the formula

$$\nu_4 = \frac{s_1\sigma^2 + s_3}{f_1\sigma^2} - (\nu_1 + \nu_2 + \nu_3). \tag{28}$$

From the first equation (15) and (25) we find the evolution equation for S_3:

$$\frac{\mathrm{d}S_3}{\mathrm{d}(2W)} = \sqrt{-\sigma^2 R(S_3)}. \tag{29}$$

The variable S_3 is real and because of (4) can oscillate only between two resolvent's zeros greater than unity. The ν_i are real if the zeros λ_i of $P(\lambda)$ consist of two complex conjugate pairs

$$\lambda_1 = \alpha + \mathrm{i}\gamma \qquad \lambda_2 = \beta + \mathrm{i}\delta \qquad \lambda_3 = \alpha - \mathrm{i}\gamma \qquad \lambda_4 = \beta - \mathrm{i}\delta. \tag{30}$$

In figure 1 the plots of ν_i, $i = 1, 2, 3$ (ν_4 is located much above ν_1), as functions of σ^2 are shown in the case of $\lambda_1 = 1 + \mathrm{i}$, $\lambda_2 = 2 + 2\mathrm{i}$. As we see, the resolvent's zeros are ordered according to $-1 < \nu_3 < \nu_2 < 1 < \nu_1 < \nu_4$ and S_3 oscillates in the interval

$$1 < \nu_1 \leqslant S_3 \leqslant \nu_4 \tag{31}$$

where $R(S_3) \leqslant 0$.

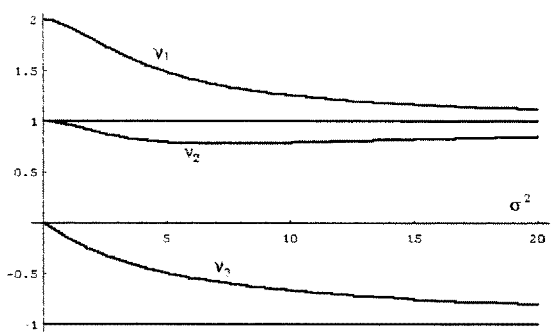

Figure 1. Dependence of the resolvent's zeros ν_i, $i = 1, 2, 3$, on the parameter σ^2 ($\lambda_1 = \lambda_3^* = 1 + \mathrm{i}$, $\lambda_2 = \lambda_4^* = 2 + 2\mathrm{i}$). For other values of λ_i the curves are deformed but their ordering remains the same.

Equations (20) and (29) permit us to calculate the period T in two ways

$$T = \frac{1}{2}\oint \frac{\mathrm{d}\mu}{\sqrt{-P(\mu)}} = \int_{\nu_1}^{\nu_4} \frac{\mathrm{d}\nu}{\sqrt{-\sigma^2 R(\nu)}}$$

which leads to useful relations

$$m = \frac{(\nu_2 - \nu_3)(\nu_4 - \nu_1)}{(\nu_1 - \nu_3)(\nu_4 - \nu_2)} = \frac{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}{(\lambda_1 - \lambda_4)(\lambda_2 - \lambda_3)} \tag{32}$$

$$\begin{aligned} \sigma^2(\nu_1 - \nu_3)(\nu_4 - \nu_2) &= (\lambda_1 - \lambda_4)(\lambda_3 - \lambda_2) = (\alpha - \beta)^2 + (\gamma + \delta)^2 \\ \sigma^2(\nu_4 - \nu_1)(\nu_2 - \nu_3) &= (\lambda_1 - \lambda_3)(\lambda_4 - \lambda_2) = 4\gamma\delta. \end{aligned} \tag{33}$$

Periodic solution of (29) gives us the desired equation for S_3

$$S_3 = \frac{(\nu_1 - \nu_3)\nu_4 + (\nu_4 - \nu_1)\nu_3 \sin^2(\sqrt{\sigma^2(\nu_1 - \nu_3)(\nu_4 - \nu_2)}W, m)}{\nu_1 - \nu_3 + (\nu_4 - \nu_1)\sin^2(\sqrt{\sigma^2(\nu_1 - \nu_3)(\nu_4 - \nu_2)}W, m)} \tag{34}$$

where the initial phase is equal to zero.

Let us now calculate S_+. Inserting (25) and (29) into (24) yields

$$\tilde{S}_+ = \sqrt{1 - S_3^2}\exp\left[\mathrm{i}\int_0^W \frac{s_1 S_3 - 2f_1}{1 - S_3^2}\,\mathrm{d}W\right]. \tag{35}$$

It is convenient to use the Weierstrass functions

$$\sin^2\left(\sqrt{\sigma^2(\nu_1 - \nu_3)(\nu_4 - \nu_2)}W, m\right) = \frac{e_1 - e_3}{\wp(W) - e_3}$$

where

$$\begin{aligned} e_1 &= -s_2/3 + \sigma^2(\nu_1\nu_4 + \nu_2\nu_3) \\ e_2 &= -s_2/3 + \sigma^2(\nu_1\nu_3 + \nu_2\nu_4) \\ e_3 &= -s_2/3 + \sigma^2(\nu_1\nu_2 + \nu_3\nu_4) \end{aligned} \tag{36}$$

the expression under the integral sign in (35) can be written as follows:

$$\frac{s_1 S_3 - 2f_1}{1 - S_3^2} = \frac{s_1 - 2f_1}{2(1 - \nu_4)}\frac{\wp(W) - \wp(\rho)}{\wp(W) - \wp(\kappa)} - \frac{s_1 + 2f_1}{2(1 + \nu_4)}\frac{\wp(W) - \wp(\rho)}{\wp(W) - \wp(\tilde{\kappa})}$$

where $\rho, \kappa, \tilde{\kappa}$ are determined by

$$\begin{aligned} \wp(\rho) &= e_3 - \sigma^2(\nu_4 - \nu_2)(\nu_4 - \nu_1) \\ \wp(\kappa) &= e_3 - \frac{\sigma^2(\nu_4 - \nu_2)(\nu_4 - \nu_1)(1 - \nu_3)}{1 - \nu_4} \\ \wp(\tilde{\kappa}) &= e_3 - \frac{\sigma^2(\nu_4 - \nu_1)(\nu_4 - \nu_2)(1 + \nu_3)}{1 + \nu_4}. \end{aligned} \tag{37}$$

Integration can be performed by means of the formula

$$\int_0^W \frac{\wp(W) - \wp(\rho)}{\wp(W) - \wp(\kappa)}\,\mathrm{d}W = W + \frac{\wp(\rho) - \wp(\kappa)}{\wp'(\kappa)}\left[\ln\frac{\sigma(\kappa + W)}{\sigma(\kappa - W)} - 2\zeta(\kappa)W\right]$$

where ζ and σ are the Weierstrass functions. As a final result we have

$$\begin{aligned} S_+ = -\sqrt{\nu_4^2 - 1}\exp\Bigg\{ &-\frac{2\mathrm{i}}{V}\left[4f_1\sigma^2 + \frac{(s_3 + s_1\sigma^2)\Delta}{f_1}\right]\xi + \frac{\mathrm{i}(s_1\nu_4 - 2f_1)}{1 - \nu_4^2}W \\ &- (\zeta(\kappa) + \zeta(\tilde{\kappa}))W\Bigg\}\frac{\sigma(W + \kappa)\sigma(W + \tilde{\kappa})\sigma^2(\rho)}{\sigma(\kappa)\sigma(\tilde{\kappa})\sigma(W + \rho)\sigma(W - \rho)} \end{aligned} \tag{38}$$

$$W = \tau - \frac{\xi}{V} \qquad V = 4\sqrt{P\left(-\tfrac{1}{2}\Delta\right)} = 4\sqrt{\left[\left(\alpha + \tfrac{1}{2}\Delta\right)^2 + \gamma^2\right]\left[\left(\beta + \tfrac{1}{2}\Delta\right)^2 + \delta^2\right]}.$$

The formulae (34) and (38) give the general solution for $\boldsymbol{S}$. The components of vector $\boldsymbol{R}$ can be found with the help of (18); in particular, we have

$$R_3 = -\frac{4}{V}f\left(-\tfrac{1}{2}\Delta\right) = -\frac{1}{V}\left(-S_3 + 2f_1\Delta + \frac{2}{f_1}(s_3 + s_1\sigma^2)\right). \tag{39}$$

2.2. The soliton limit case

Let us consider the soliton limit of this solution, i.e. when we have

$$\lambda_1 = \lambda_2 = \alpha + \mathrm{i}\gamma \qquad \lambda_3 = \lambda_4 = \alpha - \mathrm{i}\gamma.$$

Then $s_1 = 4\alpha$, $s_3 = 4\alpha(\alpha^2 + \gamma^2)$, $f_1 = 2\alpha$ and (39) gives

$$1 + R_3 = \frac{1}{V}(S_3 - 1) \tag{40}$$

where the soliton velocity equals

$$V = 4\left[\left(\alpha + \tfrac{1}{2}\Delta\right)^2 + \gamma^2\right]. \tag{41}$$

The general formulae (27), (28) for the resolvent's zeros reduce to

$$\nu_1 = \nu_2 = 1 \qquad \nu_3 = \frac{\lambda\lambda' + \lambda^*\lambda'^*}{\lambda'\lambda^* + \lambda\lambda'^*} \qquad \nu_4 = \frac{1}{\sigma^2}(\lambda\lambda^* + \lambda'\lambda'^*). \tag{42}$$

Taking into account (see equation (35))

$$(\nu_4 - 1)(1 - \nu_3) = \frac{4\gamma^2}{\sigma^2}$$

$$\sum \nu_i = 2 + \nu_3 + \nu_4 = \frac{s_3 + s_1\sigma^2}{f_1\sigma^2} = 2\frac{\alpha^2 + \gamma^2}{\sigma^2} + 2$$

we find that $(1 + \nu_3)$ and $(\nu_4 - 1)$ are the roots of a simple quadratic equation which gives

$$\begin{aligned} \nu_3 &= \frac{1}{\sigma^2}\left(\alpha^2 + \gamma^2 - \sqrt{(\alpha^2 + \gamma^2 + \sigma^2)^2 - 4\gamma^2\sigma^2}\right) \\ \nu_4 &= \frac{1}{\sigma^2}\left(\alpha^2 + \gamma^2 + \sqrt{(\alpha^2 + \gamma^2 + \sigma^2)^2 - 4\gamma^2\sigma^2}\right) \end{aligned} \tag{43}$$

which agree with (42).

Equation (34) takes the form

$$S_3 = \frac{(\nu_4 - \nu_3)\cosh^2(2\gamma W) - \nu_3(\nu_4 - 1)}{(\nu_4 - \nu_3)\cosh^2(2\gamma W) - (\nu_4 - 1)}$$

which gives

$$S_3 - 1 = 2\frac{(\nu_4 - 1)(1 - \nu_3)/(\nu_4 - \nu_3)}{\cosh(4\gamma W) - (\nu_3 + \nu_4 - 2)/(\nu_4 - \nu_3)}. \tag{44}$$

Let us introduce the parameter ϑ according to

$$\tan 2\vartheta = \frac{2\sigma\gamma}{\sigma^2 - \alpha^2 - \gamma^2} \tag{45}$$

so that

$$S_3 - 1 = V(1 + R_3) = \frac{2\gamma}{\sigma}\frac{\sin 2\vartheta}{\cosh(4\gamma W) + \cos 2\vartheta}. \tag{46}$$

Expressions (45) and (46) coincide with the Steudel soliton solution [13].

As one more particular case let us consider the wave with $\delta = 0$. The behaviour of the solution depends now on the value of the parameter σ^2. In order to show it, first take

$\alpha = 0$, so that $\lambda_1 = \lambda_3^* = \mathrm{i}\gamma$, $\lambda_2 = \lambda_4 = \beta$. Then the resolvent zeros are equal to

$$\begin{aligned}
&\text{(a)}\quad \nu_1 = \nu_4 = \frac{\sqrt{\sigma^2+\gamma^2}}{\sigma} \qquad \nu_2 = -\nu_3 = \frac{\sqrt{\sigma^2-\beta^2}}{\sigma} \qquad \sigma^2 > \gamma^2 \\
&\text{(b)}\quad \nu_1 = \frac{\beta\sqrt{\gamma^2+\sigma^2}-\gamma\sqrt{\sigma^2-\beta^2}}{\sigma^2} \qquad \nu_2 = \nu_3 = 0 \\
&\qquad\ \ \nu_4 = \frac{\beta\sqrt{\gamma^2+\sigma^2}+\gamma\sqrt{\sigma^2-\beta^2}}{\sigma^2} \qquad 0 < \sigma^2 < \beta^2.
\end{aligned} \tag{47}$$

As we see, at $\sigma^2 > \beta^2$ the zeros ν_1 and ν_4, coincide with each other which leads to a wave with constant amplitude. The corresponding solution has the form

$$\begin{aligned}
&S_3 = \frac{1}{\sigma}\sqrt{\sigma^2+\gamma^2} \qquad S_+ = \frac{\gamma}{\sigma}\exp\left(-\mathrm{i}\frac{2\sigma\sqrt{\sigma^2+\gamma^2}}{\sqrt{\frac{1}{4}\Delta^2+\gamma^2}}\xi\right) \\
&R_3 = -\frac{\Delta\sqrt{\sigma^2+\gamma^2}}{2\sigma\sqrt{\frac{1}{4}\Delta^2+\gamma^2}} \qquad R_+ = \frac{\gamma}{2\sigma\sqrt{\frac{1}{4}\Delta^2+\gamma^2}}\exp\left(-\mathrm{i}\frac{2\sigma\sqrt{\sigma^2+\gamma^2}}{\sqrt{\frac{1}{4}\Delta^2+\gamma^2}}\xi\right).
\end{aligned} \tag{48}$$

However, at $0 < \sigma^2 < \beta^2$ the other two zeros ν_2 and ν_3 coincide, and we have a special form of the periodic solution:

$$S_3 = \frac{2\beta^2\gamma^2 + \sigma^2(\beta^2-\gamma^2)}{\sigma^2[\beta\sqrt{\sigma^2+\gamma^2} - \gamma\sqrt{\sigma^2-\beta^2}\cos(2\sqrt{\beta^2+\gamma^2}W)]} \qquad \sigma^2 < \beta^2.$$

The same behaviour takes place at $\alpha \neq 0$, as we can see from figure 2, where the dependence of the resolvent's zeros on σ^2 is shown in the case of the parameter values $\alpha = \gamma = 1$, $\beta = 2$, $\delta = 0$. These curves can be considered as deformations of the curves in figure 1, when we pass from $\delta = 2$ to $\delta = 0$. Again the zeros ν_1, ν_4 coincide at $\sigma^2 > 4$ ($\beta^2 = 4$), but at $0 < \sigma^2 < 4$ we have $\nu_2 = \nu_3$. In the former region of values of σ^2, the periodic solution goes to a wave with constant amplitude, and in the latter region it goes to a special periodic wave with $m = 0$. It is important that in both cases the wave is expressed in terms of the complex spectrum λ_i, which leads to its modulation instability.

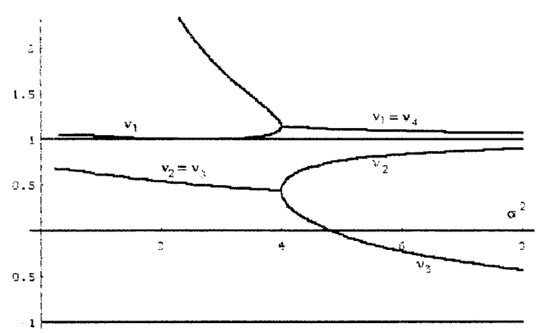

Figure 2. Dependence of the resolvent's zeros ν_i, $i = 1, 2, 3, 4$, on σ^2 for $\lambda_1 = \lambda_3^* = 1 + \mathrm{i}$, $\lambda_2 = \lambda_4 = 2$. The curves are the result of deformation of those in figure 1 when we go from $\delta = 2$ to $\delta = 0$.

3. Creation of solitons on the pulse front

The modulation of the periodic wave found above is described by the Whitham theory [20], which leads in our case to the diagonal form of the Whitham equations for the Riemann invariants λ_i, $i = 1, 2, 3, 4$. They are complex which means that the wave will have a modulation instability. It can be shown directly for the particular case (48). Indeed, the dispersion relation for small modulations of (48) has the form

$$K(\Omega) = \frac{\Omega(\sqrt{\Omega^2 - 4\gamma^2} - \Delta)}{\sqrt{\Delta^2 + 4\gamma^2}[\Omega^2 - (\Delta^2 + 4\gamma^2)]} \tag{49}$$

where K and Ω are the wavenumber and the frequency of a small modulation, respectively. We see that the solution (48) is unstable with respect to modulation with frequencies $\Omega < 2\gamma$. This modulation instability leads to the growth of any disturbance with harmonics $\Omega < 2\gamma$. In particular, the sharp front transforms into a non-uniform expanding region, one edge of which corresponds to solitons and the other one to a wave of small modulation propagating along the pulse with some group velocity. The whole region can be described as a modulated nonlinear periodic wave in which the parameters λ_i, $i = 1, 2, 3, 4$, are slow functions of ξ and τ. Averaging over fast oscillations gives the Whitham equations for λ_i, which prove to be their Riemann invariants. The derivation of these equations is similar to that of [2, 6, 11, 21]. Therefore here we shall write the final result. The Whitham equations for λ_i have the diagonal form

$$\frac{\partial \lambda_i}{\partial \xi} + \frac{1}{v_i}\frac{\partial \lambda_i}{\partial \tau} = 0 \qquad i = 1, 2, 3, 4 \tag{50}$$

where the group velocities are equal to

$$\frac{1}{v_i} = \left(1 - \frac{T}{\partial_i T}\partial_i\right)\frac{1}{V} \qquad \partial_i \equiv \frac{\partial}{\partial \lambda_i} \qquad i = 1, 2, 3, 4 \tag{51}$$

with period T being given by

$$T = \frac{1}{2}\oint \frac{\mathrm{d}\mu}{\sqrt{-P(\mu)}} = \frac{2\mathrm{K}(m)}{\sqrt{(\lambda_1 - \lambda_4 t)(\lambda_3 - \lambda_2)}} \tag{52}$$

where $\mathrm{K}(m)$ is the complete elliptic integral of the first kind and V is defined in (38). (Note that these equations can be obtained from the analogous equations for the self-induced transparency case [11] by means of replacement $\Delta \to -\frac{1}{2}\Delta$; see also [22].)

Let us consider the problem of evolution of the initially step-like pulse:

$$S_3 = \nu_4 \quad \text{at } \xi \geqslant 0 \qquad S_3 = 1 \quad \text{at } \xi < 0 \tag{53}$$

where ν_4 corresponds to the values $\lambda_1 = \lambda_3^* = \alpha + \mathrm{i}\gamma$, $\lambda_2 = \lambda_4 = \beta$, i.e. to the limit of zero modulation ($\delta = 0$) which takes place for $\sigma^2 > \beta^2$. Thus, we suggest that β^2 is less than σ^2, which corresponds to a strong Stark effect. It is important that the solution with constant amplitude with $\sigma^2 > \beta^2$ does not depend on β, until β satisfies the above inequality, so that the matching condition for β at the edge with $\delta = 0$ is fulfilled automatically. In the problem under consideration there is no characteristic dimension, hence the parameters λ_i depend only on the self-similar variable $\zeta = \xi/\tau$. Since $\lambda_3 = \lambda_1^*$ and $\lambda_4 = \lambda_2^*$, it is sufficient to use only two Whitham equations (50), which in our self-similar case take the form

$$\frac{\mathrm{d}\lambda_1}{\mathrm{d}\zeta}(v_1 - \zeta) = 0 \qquad \frac{\mathrm{d}\lambda_2}{\mathrm{d}\zeta}(v_2 - \zeta) = 0. \tag{54}$$

As we shall see, the solution corresponding to our initial data (53) is $\lambda_1 = $ constant, $v_2 = \zeta = \xi/\tau$ or

$$\alpha + \mathrm{i}\gamma = \text{constant} \tag{55}$$

$$\frac{1}{4\sqrt{((\alpha+\frac{1}{2}\Delta)^2+\gamma^2)((\beta+\frac{1}{2}\Delta)^2+\delta^2)}}\left\{1-\frac{1}{\beta-\delta+\mathrm{i}\delta}\right.$$
$$\left.\times\frac{2\mathrm{i}\delta[\alpha-\beta+\mathrm{i}(\gamma-\delta)]\mathrm{K}(m)}{[\alpha-\beta+\mathrm{i}(\gamma-\delta)]\mathrm{K}(m)-[\alpha-\beta+\mathrm{i}(\gamma+\delta)]\mathrm{E}(m)}\right\}=\frac{\tau}{\xi} \tag{56}$$

where $\mathrm{E}(m)$ is the complete elliptic integral of the second kind. On separating real and imaginary parts in the above equation, we obtain

$$\frac{\mathrm{E}(m)}{\mathrm{K}(m)}=\frac{\beta(\alpha^2+\beta^2+\gamma^2+\delta^2)-2\beta(\alpha\beta+\gamma\delta)+\frac{1}{2}\Delta[(\alpha-\beta)^2+(\gamma-\delta)^2]}{\beta(\alpha^2+\beta^2+\gamma^2+\delta^2)-2\alpha(\beta^2+\delta^2)+\frac{1}{2}\Delta[(\alpha-\beta)^2+\gamma^2-\delta^2]} \tag{57}$$

$$-\frac{1}{4\sqrt{((\alpha+\frac{1}{2}\Delta)^2+\gamma^2)((\beta+\frac{1}{2}\Delta)^2+\delta^2)}}$$
$$\times\frac{\alpha(\beta^2+\delta^2)-\beta(\alpha^2+\gamma^2)-\frac{1}{2}\Delta(\alpha^2-\beta^2+\gamma^2-\delta^2)-\frac{1}{4}\Delta^2(\alpha-\beta)}{(\alpha-\beta)[(\beta+\frac{1}{2}\Delta)^2+\delta^2]}=\frac{\tau}{\xi} \tag{58}$$

which, together with $\alpha = $ constant, $\gamma = $ constant and (32), determine implicitly the dependence of β and δ on $\zeta = \xi/\tau$.

It is convenient to express β and δ as functions of m (see [3, 5, 6]):

$$\beta=-\tfrac{1}{2}\Delta+\frac{\alpha+\frac{1}{2}\Delta}{(\alpha+\frac{1}{2}\Delta)^2+\gamma^2m^2A^2(m)}\Big((\alpha+\tfrac{1}{2}\Delta)^2+(2-m)\gamma^2A(m)$$
$$+\gamma\sqrt{4(\alpha+\tfrac{1}{2}\Delta)^2A(m)+4\gamma^2A^2(m)(1-m)-(\alpha-\tfrac{1}{2}\Delta)^2(1+mA(m))^2}\Big) \tag{59}$$

$$\delta=\frac{\gamma}{\alpha+\frac{1}{2}\Delta}mA(m)\left(\beta+\tfrac{1}{2}\Delta\right) \tag{60}$$

where we have introduced the function

$$A(m)=\frac{(2-m)\mathrm{E}(m)-2(1-m)\mathrm{K}(m)}{m^2\mathrm{E}(m)}. \tag{61}$$

In figure 3 the curves are shown along which the Riemann invariants λ_2 and λ_4 move with change of m ($\alpha = 1$, $\gamma = 1$, $\Delta = 4$). The pair of complex Riemann invariants arises at $\lambda_2 = \lambda_4 = \beta = 2.0$ (where $m = 0$, $\delta = 0$) on the real axis and after that they move in the complex plane until they coalesce with the constant pair $\lambda_1 = 1 + \mathrm{i}$ and $\lambda_3 = 1 - \mathrm{i}$ at $m = 1$. Substitution of the above expressions for β and δ into (58) gives us the dependence of m on $\zeta = \xi/\tau$. An example of such a plot is shown in figure 4. Let us investigate this region of fast oscillations at both its edges.

If $m \to 1$ we have

$$\beta+\tfrac{1}{2}\Delta\simeq\left(\alpha+\tfrac{1}{2}\Delta\right)\left(1+\frac{2\gamma\sqrt{1-m}}{\sqrt{(\alpha+\frac{1}{2}\Delta)^2+\gamma^2}}\right)\qquad\delta\simeq\gamma\left(1+\frac{2\gamma\sqrt{1-m}}{\sqrt{(\alpha+\frac{1}{2}\Delta)^2+\gamma^2}}\right)$$

and according to (58) this edge moves with the soliton velocity

$$v_{\mathrm{s}}=\left.\frac{\xi}{\tau}\right|_{m\to1}=4((\alpha+\tfrac{1}{2}\Delta)^2+\gamma^2). \tag{62}$$

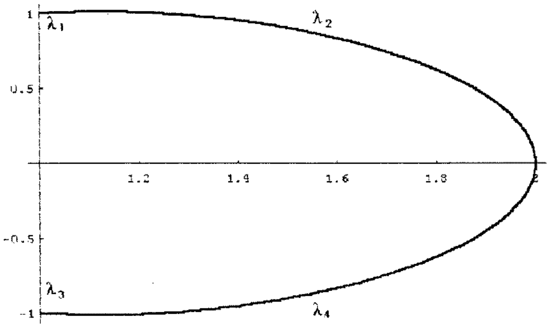

Figure 3. The paths of λ_2 and λ_4 on the complex plane λ corresponding to the self-similar solution under consideration.

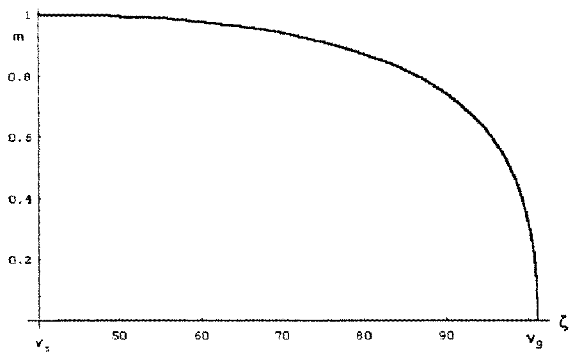

Figure 4. Dependence of the parameter m of elliptic functions on $\zeta = \xi/\tau$ for $\alpha = \gamma = 1$, $\Delta = 4$. The minimal velocity at $m \to 1$ corresponds to the soliton velocity (62) ($v_s = 40.03$ for the chosen values of parameters), and the maximal velocity at $m \to 0$ corresponds to the group velocity $v_g = d\Omega/dK$ of small modulations ($v_g = 101.2$ in our case).

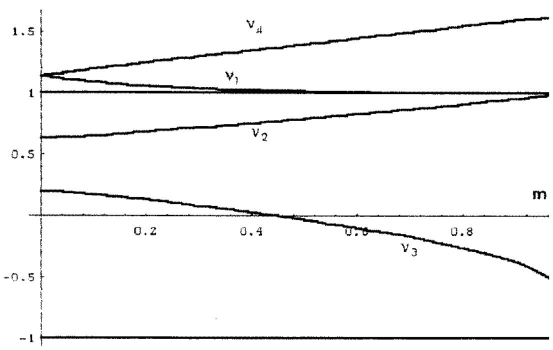

Figure 5. Dependence of the resolvent's zeros $\nu_1, \nu_2, \nu_3, \nu_4$ on m. The boundary with plane wave corresponds to $\nu_1 = \nu_4$, and the soliton boundary corresponds to $\nu_1 = \nu_2$.

If $m \to 0$, then β and δ go to the values

$$\beta = -\tfrac{1}{2}\Delta + \left(\alpha + \tfrac{1}{2}\Delta\right)\left[1 + \frac{3\gamma^2}{4(\alpha + \frac{1}{2}\Delta)^2}\left(1 + \sqrt{1 + \frac{8(\alpha + \frac{1}{2}\Delta)^2}{9\gamma^2}}\right)\right] \qquad (63)$$

$$\delta = 0$$

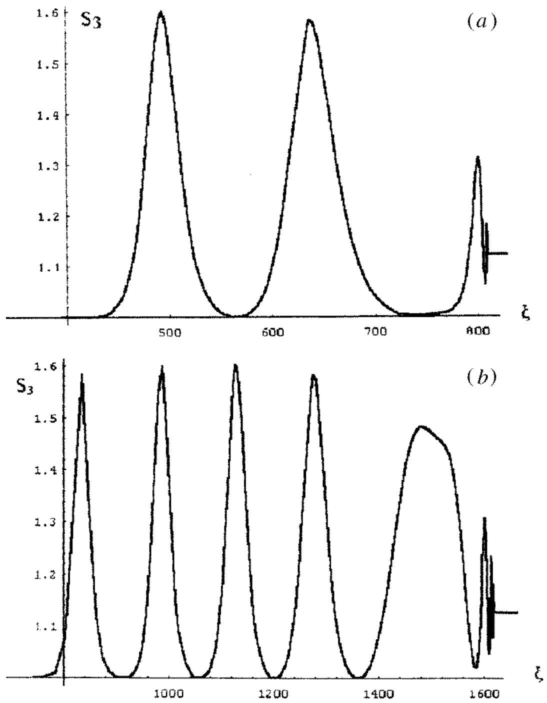

Figure 6. Dependence of S_3 on the space coordinate ξ at two moments of time: (a) $\tau = 8$, (b) $\tau = 16$. The calculation was done with the use of (34), where $\alpha = 1$, $\gamma = 1$, β and δ depend on m according to (59) and (60), and ζ depends on m according to (58).

and (58) becomes

$$\frac{1}{v} = \frac{\tau}{\xi} = \frac{\alpha^2 + \gamma^2 - \alpha\beta + \frac{1}{2}\Delta(\alpha - \beta)}{4(\beta + \frac{1}{2}\Delta)^2(\alpha - \beta)\sqrt{(\alpha + \frac{1}{2}\Delta)^2 + \gamma^2}}. \tag{64}$$

In this limit of small modulation the Whitham theory must reproduce the linear approximation, that is v must coincide with the corresponding group velocity of the modulation wave. From the general periodic solution (34) and (38) we know that the phase of the modulation wave at $\delta = 0$ has the form

$$\sqrt{(\alpha - \beta)^2 + \gamma^2}\left(\tau - \frac{\xi}{4(\beta + \frac{1}{2}\Delta)\sqrt{(\alpha + \frac{1}{2}\Delta)^2 + \gamma^2}}\right)$$

that is the frequency Ω and the wavenumber K of the modulation wave are expressed in terms of the parameters α, β, γ as follows:

$$\Omega = 2\sqrt{(\alpha - \beta)^2 + \gamma^2} \qquad K = \frac{\Omega}{4(\beta + \frac{1}{2}\Delta)\sqrt{(\alpha + \frac{1}{2}\Delta)^2 + \gamma^2}}. \tag{65}$$

It is easy to check that these values satisfy the generalization of the dispersion relation (49) on $\alpha \neq 0$:

$$K(\Omega) = \frac{\Omega(\sqrt{\Omega^2 - 4\gamma^2} - 2(\alpha + \frac{1}{2}\Delta))}{2\sqrt{(\alpha + \frac{1}{2}\Delta)^2 + \gamma^2}[\Omega^2 - 4((\alpha + \frac{1}{2}\Delta)^2 + \gamma^2)]}. \tag{66}$$

Calculation of group velocity $v_g = (dK/d\Omega)^{-1}$ at Ω from (65) reproduces, as we expected, the solution (63) of the Whitham equations in the limit of small modulation. It can be shown that $v_g > v_s$ at all α and γ. The dependence of $\nu_1, \nu_2, \nu_3, \nu_4$ on m is shown in figure 5. At $m = 0$ (plane-wave boundary) we have $\nu_1 = \nu_4$, and at $m = 1$ (soliton limit) we have $\nu_1 = \nu_2$. This plot looks like the behaviour of real Riemann invariants in the Gurevich–Pitaevskii-type problems [7, 23–25].

We see that the sharp front transforms into the expanding oscillatory region. The slower edge of this region propagates with the soliton velocity and consists of the train of solitons. The faster edge propagates with the group velocity of the small modulation wave. The whole region can be described as a modulated nonlinear periodic solution of TPP equations. This oscillatory region is shown in figure 6 for two values of τ. The plots demonstrate the process of soliton creation on the front of the pulse.

Thus, the method discussed here gives us an efficient approach to the nonlinear theory of modulation instability and can be applied to a variety of different problems.

Acknowledgments

We are grateful to H Steudel for valuable discussions. FG thanks CNPq-Brazil for financial support.

References

[1] Menyuk C R 1989 *Phys. Rev. Lett.* **62** 2937
[2] Kamchatnov A M 1992 *Phys. Lett.* **162A** 389
[3] El' G A, Gurevich A V, Khodorovkiĭ V V and Krylov A L 1993 *Phys. Lett.* **177A** 357
[4] Bikbaev R F and Kudashev V R 1994 *Phys. Lett.* **190A** 255
[5] Kamchatnov A M 1995 *Phys. Lett.* **202A** 54
[6] Kamchatnov A M and Pavlov M V 1995 *J. Phys. A: Math. Gen.* **28** 3279
[7] Zakharov V E, Manakov S V, Novikov S P and Pitaevskiĭ L P 1990 *The Theory of Solitons* (Moscow: Nauka)
[8] Kamchatnov A M 1990 *J. Phys. A: Math. Gen.* **23** 2945
[9] Kamchatnov A M 1990 *Zh. Eksp. Teor. Fiz.* **97** 144 (Engl. transl. 1990 *Sov. Phys.–JETP* **70** 80)
[10] Kamchatnov A M 1992 *Zh. Eksp. Teor. Fiz.* **102** 1606 (Engl. transl. 1992 *Sov. Phys.–JETP* **75** 868)
[11] Kamchatnov A M and Pavlov M V 1995 *Zh. Eksp. Teor. Fiz.* **107** 44 (Engl. transl. 1995 *Sov. Phys.–JETP* **80** 22)
[12] Maimistov A I, Basharov A M, Elyutin S O and Sklyarov Yu M 1990 *Phys. Rep.* **191** 1
[13] Steudel H 1983 *Physica* **6D** 155
[14] Kaup D J 1983 *Physica* **6D** 143
[15] Meinel R 1984 *Opt. Commun.* **49** 224
[16] Ablowitz M J, Kaup D J, Newell A C and Segur H 1974 *Stud. Appl. Math.* **53** 241
[17] Its A R and Kotlyarov V P 1976 *DAN UkrSSR* **11** 965
[18] Kozel V A and Kotlyarov V P 1976 *DAN UkrSSR* **11** 878
[19] Forest M G and McLaughlin D W 1982 *J. Math. Phys.* **27** 1248
[20] Whitham G B 1974 *Linear and Nonlinear Waves* (New York: Wiley)
[21] Kamchatnov A M 1994 *Phys. Lett.* **186A** 387
[22] Zabolotskii A A 1994 *Phys. Rev.* A **50** 3384
[23] Gurevich A V and Pitaevskiĭ L P 1973 *Zh. Eksp. Teor. Fiz.* **65** 590 (Engl. transl. *Sov. Phys.–JETP* **38** 291)
[24] Gurevich A V, Krylov A L and El' G A 1992 *Zh. Eksp. Teor. Fiz.* **101** 1797 (Engl. transl. *Sov. Phys.–JETP* **74** 957)
[25] El' G A and Krylov A L 1995 *Phys. Lett.* **203A** 77